Optimal Selling Mechanisms with Buyer Price Search*

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Abstract

We study optimal dynamic selling mechanisms when the buyer initially and privately endowed with value of the object on sale can conduct costless search for a private second-stage outside price. The unique features of this problem include that second-stage incentive compatibility does not require the usual full monotonicity of allocation rule and off-equilibrium-path best strategy cannot be pinned down. We propose a modified Myerson convexification procedure that regularizes the buyer’s virtual value in the dimension of the outside price to identify the revenue-maximizing mechanism in this dynamic setting. At optimum, the seller simply offers a first-stage fixed price, which is only taken by high value types. In contrast, if buyer’s second stage outside price is publicly observable, the optimal selling mechanism would take the form of a fixed first stage coupled with a price matching at second period.

Keywords: Dynamic mechanism design; Outside option; Price search; Price match.

JEL Classification Numbers: D11, D82, D86.

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1 Introduction

Situations are ubiquitous in which buyers, when making their purchase decisions, can avail themselves of more favorable prices in the quite near future by search. For example, eBay hosts online car auctions regularly and frequently on such a timeline, and the Land Transport Authority of Singapore holds two auctions every month to sell certificates of entitlement (COE), which are the quota-controlled auto licences. Anticipation of a possible good outside option definitely affects buyers’ purchase decisions, because they can postpone a purchase decision and wait for a possible price cut. This in turn would influence the seller’s optimal selling strategy.

In this paper, we explore the revenue-maximizing dynamic selling mechanism for a principal who sells an indivisible object to a single buyer in a two-stage model with costless buyer search.\(^1\)\(^2\) The buyer is initially endowed with his private value in the first stage, which is his private information; he will further observe his outside option price in the second stage. Types of both stages are continuously distributed. We consider two scenarios by allowing the second-stage information to be either public or private.

We find that for the scenario with a public outside option, the optimal mechanism involves a cutoff value \(v^*\). The first-stage types above \(v^*\) purchase in the first stage at a fixed price; the first-stage types below \(v^*\) conduct the search and buy from the seller in the second stage if and only if the option price turns out to be lower than their values. The second-stage price is set exactly at the buyer’s option, which is essentially a price match. For the scenario with a private outside option, the optimal mechanism degenerates to a single-stage mechanism, which takes the form of a fixed-price offer at the first stage.\(^3\) There is a cutoff type \(v^{**}\), which is lower than \(v^*\). First stage types above \(v^{**}\) purchase from the

\(^1\)Considering costless search allows us to focus on studying the impact of privateness of second-stage outside price rather than the impact of search cost.

\(^2\)Our analysis adopts an environment of one buyer, which is often adopted in the literature for its own merit. For example, Armstrong and Zhou (2015) focus on a setting with one buyer. On the other hand, our analysis can be generalized to a multi-buyer environment. The single buyer environment serves as a heuristic for the analysis of multi-buyer setting.

\(^3\)Krähmer and Strausz (2015a) also find that a single-stage mechanism after second-stage information is revealed is optimal when the buyer has ex post withdrawal right, which guarantees the buyer a non-negative payoff ex post. In our setting, the optimal single-stage mechanism is instead offered at the first stage.
seller in the first stage; types below \( v^{**} \) do not purchase from the seller. Types below \( v^{**} \) conduct a search and buy from the outside option if and only if the outside option price is lower than the value. While the optimal mechanism must be implemented in two stages in the setting with public second-stage information, in the setting with private second-stage information the optimal mechanism can be implemented in a single stage.

Our findings contrast with that of Armstrong and Zhou (2015), who first study an optimal selling mechanism with strategic buyer search.\(^4\) In their setting, the privateness of the second-stage buyer information does not affect the optimal design. The divergence between our findings and theirs is due to that buyer search is modeled differently in the two studies. Our paper adopts a search model with homogeneous products in which the buyer simply searches for an optional price of a homogeneous object that the buyer values the same, while Armstrong and Zhou study a search model with differentiated products in which the buyer searches for an outside net payoff from purchasing a potentially heterogeneous object that the buyer may value differently.\(^5\) One important difference across these two models can be illustrated as follows. Consider the case with public second-stage information. In our model, if the buyer has a superior outside option (e.g., a lower option price than buyer value), the seller prefers to match the price and collect some revenue. However, in Armstrong and Zhou, the seller has no incentive to match the outside option if the buyer finds a superior outside option (e.g., an outside net payoff that is higher than the buyer’s value).

In Armstrong and Zhou (2015), the optimal mechanism under the public second-stage type is incentive compatible even when the second-stage type is the buyer’s private information, since in their model the optimal allocation rule is monotone with respect to the second-stage type. However, in our model, the optimal mechanism under the public second-stage type is not incentive compatible when the second-stage type is the buyer’s private information: The second-stage virtual value function is non-monotonic in the buyer’s outside option price. For example, with a public outside price, the seller can match it in the second stage.\(^6\) However, this is not incentive compatible when the outside option price is private, as the buyer has incentive to under-report it. The optimal second-stage allocation rule under the public second-stage type fails to satisfy a necessary semi-monotonicity condition in second stage.

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\(^4\)They allow positive buyer search cost, but their analysis also applies to costless search.

\(^5\)Wolinsky (1986) studies this environment of search with differentiated products.

\(^6\)This will be shown as a feature of the optimal mechanism when the outside option is public.
types, which is required by second-stage incentive compatibility under the private second-stage type. This semi-monotonicity condition, by and large, requires the buyer to get the object more often from the seller if his outside price is higher, which is clearly violated by the optimal allocation rule under the public-second stage option price.

Kakade, Lobel, and Nazerzadeh (2013) and Battaglini and Lamba (2019) demonstrate environments in which the irrelevance of privateness of ex post information would fail. The optimal mechanisms in these environments remain largely unsolved since the technique of considering a relaxed problem is not valid in searching for the optimal mechanisms. In our setting, the irrelevance of privateness of ex post information fail for similar reason: the virtual value function under public ex post information is not monotone in ex post types, which means the optimal allocation rule under public ex post information simply cannot be incentive compatible under private ex post information. This non-monotonicity issue adds further complications in characterizing the optimal design in our environment.

Optimal designs have been successfully characterized in many dynamic environments. To our best knowledge, typically the key for solving the optimal designs lies in that in these settings, agents’ off-equilibrium-path optimal responses upon an early stage deviation can be explicitly identified and they are not mechanism contingent. For example, agents reveal truthfully their ex post types even on the off equilibrium paths in Baron and Besanko (1984), Courty and Li (2000), Battaglini (2005), Garrett and Pavan (2012), Kakade, Lobel, and Nazerzadeh (2013), Pavan, Segal and Toikka (2014), Halac and Yared (2014), Armstrong and Zhou (2015), Krähmer and Strausz (2015a), Bergemann, Castro and Weintraub (2018), Liu and Lu (2018) and Battaglini and Lamba (2019); and agents correct their first stage lie in Esö and Szentes (2007), Krähmer and Strausz (2015b), and Bergemann and Strack (2015). However, in our environment, the above key property of off-equilibrium-path optimal responses that facilitates the characterization of optimal mechanisms no longer holds, which makes our problem unique and demanding.

Due to the above identified issues, searching for the optimal design with private second-stage information calls for an innovative procedure. We adapt the conventional procedure of considering a relaxed problem with public second-stage information by further incorporating the semi-monotonicity of the second-stage allocation rule, which is required by the second-stage incentive compatibility. With this additional constraint, even the relaxed problem with public ex post information is not easy to solve. We rely on the Myerson convexification
procedure—which transforms the original non-monotonic second-stage virtual value function to its regularized version—to identify the solution for the relaxed problem. Nevertheless, the validity of the Myerson convexification procedure still needs to be reexamined and established in a manner adapted to our context, since in our environment second-stage incentive compatibility only requires that the second-stage allocation rule be monotone when the outside price is below the first-stage type. For an outside price above the first-stage type, second-stage incentive compatibility only requires that its corresponding second-stage selling probability be higher than that of any outside price below the first-stage type.\textsuperscript{7}

Relying on an adapted Myerson convexification procedure, we are able to identify the optimal allocation rule in the relaxed problem with public ex post information and the semi-monotonicity condition required by second-stage incentive compatibility. To our best knowledge, this is the first time in the literature that a Myerson convexification procedure is successfully employed in a dynamic mechanism design environment. Solution of the relaxed problem thus renders a revenue upper bound for the original problem with private ex post information. We further identify a feasible (incentive compatible and individual rational) mechanism in the original problem that achieves the above revenue upper bound, which means that the identified mechanism must be optimal in the original problem.

Our optimal mechanism under private outside option differs qualitatively from that of Armstrong and Zhou (2015). In their mechanism, when search cost is zero,\textsuperscript{8} low-value types must go for the outside options; high-value types pay a deposit in the first stage and come back to buy from the seller if and only if their outside option turns out to be worse than a certain level. While for Armstrong and Zhou (2015) the optimal selling mechanism must involve two stages, in our setting the optimal mechanism degenerates to a single-stage one and weakly deters search by high-value types. Our optimal mechanism under a public outside option also differs qualitatively from that of Armstrong and Zhou. In our model, higher-value types directly purchase from the seller; the low-value types conduct search and buy from the seller only when their outside price is lower than their values. These divergences illustrate the impact of different models of the buyer’s outside option on the optimal selling mechanism design.

\textsuperscript{7}In the environment of Myerson (1981), the allocation rule is monotone in the players’ types over the whole type space.

\textsuperscript{8}Armstrong and Zhou assume a positive search cost, and their findings are valid even when the cost is zero.
In addition to the dynamic mechanism design literature we have cited extensively above, our paper contributes to the well-established literature on market with search. Armstrong and Zhou’s (2015) study is a most recent significant contribution. Different from our study, they adopt a search model with differentiated products, in which the buyer searches for an outside net payoff. Many studies in this literature instead adopt a search model with homogeneous products in which agents simply search for alternative prices, which is what we follow in our setup. Stahl (1989) studies an oligopoly search model with homogeneous products. Zhu (2012) offers a dynamic model of opaque over-the-counter markets with seller search for an attractive price. Ellison and Wolitzky (2012) study the use of an obfuscation strategy by oligopolistic firms to deter consumer search on prices. Janssen and Parakhonyak (2014) study market competition with consumer search on prices and costly revisits. Che et al. (2019) study a seller’s strategy of using deposits to deter buyers’ price search in auction environments. Our study is most closely related to Armstrong and Zhou, who study the seller’s optimal selling mechanism in addition to conventional search-deterring sale techniques such as “buy-now” discounts and exploding offers, etc. We differentiate from Armstrong and Zhou by considering a seller-optimal mechanism with buyer search on prices and allowing for multiple buyers. These modeling differences generate both significant technical challenges in identifying the optimal mechanisms when ex post information is private and markedly different implications for the optimal selling mechanism.

The rest of this paper is organized as follows. In Section 2, we set up the model. In Section 3, we derive the optimal two-stage mechanisms. Section 4 concludes. The Appendix collects the technical proofs.

2 The Model

We consider an environment in which there is one seller who has an indivisible object (e.g., a laptop computer) to sell to one risk-neutral buyer. The seller’s own valuation for the object is normalized to zero. There are two stages. At stage 1, the buyer is endowed with private information about his valuation $v$ for the object. The buyer’s valuation remains the same across the two stages. We assume that $v \in V := [0, \bar{v}]$ is randomly distributed following cumulative distribution function $F(\cdot)$ with continuous and strictly positive density $f(\cdot)$. We further assume monotone hazard rate, that is, $f(v)/(1 - F(v))$ is weakly increasing in $v$. At
stage 2, the buyer will privately observe an outside option, which takes the form of alternative price \( \tau \) offered by a different source for the same object (e.g., a laptop of the same brand and same model). We assume that \( \tau \) is randomly distributed following cumulative distribution function \( G(\cdot) \) with continuous and strictly positive density \( g(\cdot) \) on support \( V \). The buyer has a single unit of demand. He can obtain the object from either the seller or the outside option, but not both. For simplicity, we assume that the search for information \( \tau \) is costless to the buyer.\(^9\) Both the valuation distribution \( F \) and outside option price distribution \( G \) are common knowledge, and \( v \) and \( \tau \) are independent.

According to Myerson (1982), without loss of generality, the seller considers the following two-stage direct selling mechanisms:

At stage 1, the buyer who is endowed with private value \( v \) is asked to make a report \( v' \in V := [0, \bar{v}] \). Upon reporting \( v' \), the buyer obtains the object with probability \( p_1(v') \in [0, 1] \) from the seller, and pays \( x_1(v') \) to the seller. With probability \( 1 - p_1(v') \), the buyer does not get the object from the seller and the game enters the second stage.

At stage 2, the buyer who observes his outside option price \( \tau \) is asked to further make a report \( \tau' \in V := [0, \bar{\tau}] \). Upon reporting \( \tau' \), the buyer obtains the object with probability \( p_2(v', \tau') \) from the seller and pays \( x_2(v', \tau') \) to the seller. The feasibility condition requires \( p_2(v', \tau') \in [0, 1] \).

The buyer can obtain the object from either the seller or the outside option, but not both. The buyer also has the option of not buying from either source. If the buyer with value \( v \) simply waits for the outside option, his expected payoff from the outside option is \( E_\tau \max\{v - \tau, 0\} \), where \( \max\{v - \tau, 0\} \) is the value of option \( \tau \) to the buyer.

We are going to characterize the optimal two-stage selling mechanism. First, though, it will be useful for us to examine a benchmark case in which the buyer’s outside option price at the second-stage is public information. Both the seller and the buyer observe the buyer’s outside option price \( \tau \) at the second stage, and thus the second-stage mechanism can be directly defined on \( \tau \). Unlike in the original environment, in which \( \tau \) is privately known by the buyer, the seller can ignore the second-stage incentive compatibility constraint.

\(^9\)The focus of our analysis is to study the impact of the privateness of second-stage outside price rather than the impact of search cost. Armstrong and Zhou (2015) assume endogenous стратегический search frictions, while we assume costless search. Costless search has been adopted by Ellison and Wolitzky (2012) and Krähmer and Strausz (2015a). Assuming costless search allows us to focus on studying search deterrence using selling mechanisms by completely eliminating the impact of search cost.
of the buyer. In the original environment, the seller does not observe the buyer’s outside option price $\tau$; she has to elicit this information from the buyer. Clearly, the optimal solution in the benchmark environment must provide an upper bound for the revenue achievable in the original setting. An interesting question is whether this bound is reachable. We will demonstrate that the privacy of the buyer’s second-stage information renders the optimal allocation different across the two settings, and thus the bound in the benchmark setting is not achievable in the original environment.

As a result, the optimal solution in the original setting cannot be identified following a standard procedure, as in Esö and Szentes (2007) and Pavan et al. (2014). Instead, we need to identify a necessary condition for second-stage incentive compatibility in the original setting, and include this condition in the benchmark environment to obtain an upper bound of revenue. We then demonstrate that the bound is achievable by an incentive compatible mechanism in the original setting.

3 The analysis

3.1 Benchmark: Public Outside Option Price

We now look at the benchmark environment, in which the second-stage buyer’s outside option price $\tau$ is public. We first look at the second stage. Under the second-stage mechanism $\{p_2(\cdot, \cdot), x_2(\cdot, \cdot)\}$, if the buyer reports $v'$ in the first stage and his value is $v$ and his outside option price is $\tau$, then his second-stage expected payoff is

$$\tilde{\pi}(v, v', \tau) = p_2(v', \tau)v + \left(1 - p_2(v', \tau)\right)\max\{v - \tau, 0\} - x_2(v', \tau) = \max\{v - \tau, 0\} + p_2(v', \tau)\min\{v, \tau\} - x_2(v', \tau).$$

Since $\tau$ is known to the seller, there is no need to take care of incentive compatibility. We further relax the problem by ignoring individual rationality constraints in the second stage on and off the equilibrium path. Alternatively, one can treat the benchmark scenario as a single-stage design problem at the first stage, with the second-stage outcome fully specified by mechanism $\{p_2(\cdot, \cdot), x_2(\cdot, \cdot)\}$.

We now move to the first stage. Under the first-stage mechanism $\{p_1(\cdot), x_1(\cdot)\}$, the buyer’s interim expected payoff is

$$\pi(v, v') = p_1(v')v + \left(1 - p_1(v')\right)E_\tau\tilde{\pi}(v, v', \tau) - x_1(v').$$
Denote $\pi(v) = \pi(v, v)$. The first-stage incentive compatibility constraint requires that for all $v, v' \in [0, \bar{v}]$,

$$\pi(v) \geq \pi(v, v').$$

The individual rationality constraint at the first stage requires that for all $v \in [0, \bar{v}]$,

$$\pi(v) \geq E_\tau \max\{v - \tau, 0\}.$$  

(3.2)

The seller’s revenue is the payment she gathers from the buyer in two stages. Hence, under an incentive compatible mechanism $\mathcal{M} = \{p_1(\cdot), x_1(\cdot); p_2(\cdot, \cdot), x_2(\cdot, \cdot)\}$ with public second-stage information $\tau$, the seller’s expected revenue is

$$R = \int_0^\bar{v} \left\{x_1(v) + \left(1 - p_1(v)\right)E_\tau x_2(v, \tau)\right\}dF(v).$$

Then seller’s problem is to maximize $R$ subject to constraints (3.1) and (3.2).

Following the standard Myersonian procedure in a single-stage problem and setting $\pi(0) = 0$ lead to the following expression for the seller’s revenue:

$$R = \int_0^\bar{v} \left\{p_1(v)\lambda(v) + \left(1 - p_1(v)\right)\left[G(v)\lambda(v) - \int_0^v \tau dG(\tau)\right] + \int_v^\bar{v} p_2(v, \tau)\tau dG(\tau) + \int_0^\bar{v} p_2(v, \tau)\lambda(v)dG(\tau)\right\}dF(v),$$

(3.3)

which only depends on the allocation rule.\footnote{There is no loss to consider full participation in the first stage, since the seller can always set $p_1 = 0$, and $x_1 = 0$ for a nonparticipating type.}

The optimal allocation rule can thus be identified by point-wise optimization, and we can further identify a payment rule to support this allocation. The optimal selling mechanism is presented in the following Proposition 1. We would like to emphasize that the optimal mechanism derived in the above relaxed environment while ignoring second-stage individual rationality remains incentive compatible in the first stage and individual rational on the equilibrium path in both stages, and thus revenue-maximizing (achieving the same revenue) even if we allow the buyer to stay out in the second stage (upon finding out his option) following a lie in the first stage.

\textbf{Proposition 1.} Let

$$\lambda(v) = v - \frac{1 - F(v)}{f(v)}$$

\footnote{Details are relegated to the proof of Proposition 1.}
be the buyer’s virtual value, and let $v^*$ satisfy $\lambda(v^*) = 0$. If the buyer’s searched outside option price $\tau$ is public information, then the optimal mechanism is as follows:

1) if $v < v^*$, the type-$v$ buyer searches outside option price and returns to purchase from the buyer if $\tau \leq v$ at price $\tau$;

2) if $v \geq v^*$, the type-$v$ buyer purchases directly at price $E_{\tau} \min\{v^*, \tau\}$.

Proof. See Appendix. □

Proposition 1 reveals that when the second-stage information is public, the seller would sell in both stages at optimum. If the buyer’s valuation is sufficiently high, i.e., $v \geq v^*$, then the seller sells the object in the first stage at a fixed price $E_{\tau} \min\{v^*, \tau\}$. If the buyer’s valuation is low, i.e., $v < v^*$, then the object will be sold only when the buyer’s outside option price is observed to be lower than $v$, and the price equals the observed outside option price.

Intuition for the allocation rule in Proposition 1 goes as follows. Suppose instead that the seller observes a buyer’s valuation $v$. The seller can sell at the first stage by charging a price of $E_{\tau} \max\{v - \tau, 0\}$, which fully extracts the buyer’s surplus. When the buyer’s valuation $v$ is private at the first stage, Proposition 1 reveals that the types $v < v^*$ have to search and purchase from the seller at the second stage only when $\tau < v$. Only the types $v \geq v^*$ purchase from the seller at the first stage. This downward distortion in allocation is for the purpose of reducing information rent when $v$ is private.

Surprisingly, instead of allocating the object to the buyer when the outside option is inferior, say $\tau > v$, and charging a price as high as $v$ at the second stage, the mechanism allocates the object to the buyer when the outside option turns out to be superior and charges a price lower than $v$. It looks as if the seller is not adopting an optimal selling strategy. However, this is not the case. If the seller commits to sell to a buyer with an inferior outside option at a price that equals his reported value, then the buyer with valuation lower than $v^*$ would have an incentive to under-report at the first stage to gain a better chance of obtaining the object at a lower price. Clearly, the buyer would report zero at optimum. As a result, the seller cannot collect any payment from the buyer at the second stage. Therefore, by selling only to the buyer with a superior outside option at the second stage, the seller prevents a low-valuation buyer’s deviation in the first stage, which ensures that seller can at least collect some payment when the buyer’s option is superior.
3.2 Private Information Outside Option Price

We now turn to the original environment in which both valuation and outside option price are privately observed by the buyer. Suppose that $v'$ is reported in the first stage, while the true valuation is $v$. As in the benchmark setting, we start by ignoring the second-stage individual rationality constraint on and off the equilibrium path. We will eventually show that this relaxation does not affect the optimal solution. Since the second-stage information is privately observed by the buyer, the seller has to ask the buyer to report it. Assume that the outside option price is $\tau$ and the buyer reports $\tau'$, then his second-stage expected payoff is

$$\tilde{\pi}(v, v', \tau, \tau') = p_2(v', \tau')v + \left(1 - p_2(v', \tau')\right)\max\{v - \tau, 0\} - x_2(v', \tau')$$

$$= \max\{v - \tau, 0\} + p_2(v', \tau')\min\{v, \tau\} - x_2(v', \tau').$$

If the buyer is truth-telling at both stages, i.e., $v' = v$ and $\tau' = \tau$, then his expected payoff is written as

$$\tilde{\pi}(v, \tau) = \max\{v - \tau, 0\} + p_2(v, \tau)\min\{v, \tau\} - x_2(v, \tau).$$

When the first stage is truthful, second-stage incentive compatibility requires that

$$\tilde{\pi}(v, \tau) \geq \tilde{\pi}(v, \tau, \tau'), \forall v, \tau, \tau'. \quad (3.4)$$

The following lemma provides a necessary condition for a two-stage mechanism to be incentive compatible in the second stage after a truthful first stage.

**Lemma 1.** If the object is not sold in the first stage with a truthful report, incentive compatibility in the second stage requires the following conditions:

$$\text{if } 0 \leq \tau' < v, \tau' < \tau, \text{ then } p_2(v, \tau') \leq p_2(v, \tau). \quad (3.5)$$

**Proof.** See Appendix. \qed

From Lemma 1, incentive compatibility in two stages must mean that the second-stage allocation rule $p_2(v, \tau)$ is weakly increasing in $\tau$ when $\tau \leq v$. If $\tau \geq v$, while it is not clear whether $p_2(v, \tau)$ is monotonic in $\tau$ or not, $p_2(v, \tau)$ should be at least as large as $p_2(v, v)$. Figure 3.1 depicts such an allocation rule.
In the benchmark case with public $\tau$, the optimal mechanism is presented in Proposition 1. It is clear that the mechanism does not satisfy the Lemma 1 semi-monotonicity. Therefore, it cannot be the optimal mechanism when $\tau$ is private.

In the sequential screening environments in which optimal designs have been successfully characterized, typically agents’ off-equilibrium-path optimal responses upon an early stage deviation can be explicitly identified and they are not mechanism contingent. For example, agents reveal truthfully their ex post types even on the off equilibrium paths in Baron and Besanko (1984) and many others; and agents correct their first stage lie in Esö and Szentes (2007) and many others.\(^{12}\) However, in our environment, this key property that greatly facilitates the characterization of optimal mechanisms no longer holds due to possible non-monotonicity of second stage allocation rules that is illustrated in Lemma 1. This complication makes our problem unique and demanding. We need to discover a new procedure to search for the optimal design.

The way we solve for the optimal mechanism with private $\tau$ is to identify a revenue bound in the benchmark environment while imposing the semi-monotonicity condition (3.5). We then identify an incentive compatible and individual rational mechanism in the original two-stage environment with private $\tau$, which achieves the revenue bound.

\(^{12}\)A more complete list is provided in the introduction.
Define
\[ u(v, \tau) = \begin{cases} 
\tau & \text{if } \tau \leq v, \\
\lambda(v) & \text{otherwise},
\end{cases} \]
which we interpret as the buyer’s second-stage virtual value.

Using \( u(v, \tau) \), the seller’s expected revenue (3.3) in the benchmark setting with public \( \tau \) can be alternatively written as
\[
R = \int_0^v \left\{ p_1(v) \lambda(v) + \left(1 - p_1(v)\right) \left[G(v) \lambda(v) - \int_0^\tau \tau dG(\tau) + \int_0^\tau p_2(v, \tau) u(v, \tau) dG(\tau)\right]\right\} dF(v).
\]
(3.6)

We will identify the allocation rule that maximizes the above revenue expression subject to semi-monotonicity condition (3.5), which would provide a revenue bound for the seller’s problem in the original setting with private \( \tau \). We then show that the bound is indeed achievable.

For this purpose, for given valuation \( v \), we first identify \( p_2(v, \tau) \), which maximizes \( \int_0^\tau p_2(v, \tau) u(v, \tau) dG(\tau) \) subject to constraint (3.5). Note that \( u(v, \tau) \), however, is not monotone in \( \tau \), since \( \lambda(v) < v \) for \( v < \bar{v} \). Thus the problem of finding \( p_2(v, \tau) \) is irregular.
Following the convexification procedure in Myerson (1981), we first consider \( u(v, \tau) \) as a function of \( \tau \) and then establish the corresponding convex hull, which would render the regularized \( \tilde{u}(v, \tau) \) that is monotone increasing in \( \tau \). Therefore the maximization problem \( \int_0^\tau p_2(v, \tau) \tilde{u}(v, \tau) dG(\tau) \) is regular, which can be solved by point-wise optimization. We will show that the solution to the regular convexified problem is the solution to the original one. It, however, must be established in a more sophisticated way than Myerson (1981), since in our setting \( p_2(v, \tau) \) might not be monotone in \( \tau \) when \( \tau > v \). We consider this methodological innovation as a technical contribution to Myerson’s convexification procedure in mechanism design.

**Convexification** Since distribution function \( G \) is continuous and strictly increasing, its inverse \( G^{-1} \) is also continuous and strictly increasing. Let \( q = G(\tau) \). Then for any \( q \in [0, 1] \), let
\[
h(q) = u(v, G^{-1}(q)) = \begin{cases} 
G^{-1}(q) & \text{if } q \leq G(v), \\
\lambda(v) & \text{if } q > G(v),
\end{cases}
\]
and let
\[
H(q) = \int_0^q h(s) ds = \begin{cases} 
\int_0^q G^{-1}(s) ds & \text{if } q \leq G(v), \\
\int_0^G(v) G^{-1}(s) ds + \lambda(v)(q - G(v)) & \text{if } q > G(v).
\end{cases}
\]
If $q \leq G(v)$, we have $H'(q) = G^{-1}(q) \geq 0$ and $H''(q) = 1/g(G^{-1}(q)) > 0$, which implies that function $H(q)$ is convex on this part. If $q > G(v)$, $H(q)$ is linear and the slope is $\lambda(v)$. Since $v > \lambda(v)$, function $H(q)$ is not differentiable at $q = G(v)$ and the slope is smaller on the left side of $G(v)$. At the end point $q = 1$, $H(1)$ equals $\int_0^\bar{v} u(v, \tau)dG(\tau)$, which merely depends on valuation $v$.

Let $\tilde{H} : [0, 1] \to \mathbb{R}$ be the convex hull of $H(q)$. $\tilde{H}(q)$ is continuously differentiable as depicted in the following three graphs. Depending on the value of $v$, there are three cases, which correspond, respectively, to Figures 3.2, 3.3, and 3.4. In Cases 1 and 2, we have $H(1) = \int_0^\bar{v} u(v, \tau)dG(\tau) > 0$. The $\tilde{H}$ coincides with $H$ below a threshold of $q$ and then becomes a straight line ending at the same point with $H$. In Case 3, $H(1) = \int_0^\bar{v} u(v, \tau)dG(\tau) \leq 0$, and $\tilde{H}$ is a straight line linking the origin and the end point of $H$. By the convexification process, we have following results.

**Lemma 2.** Let $\Delta H(\tau) = H(G(\tau)) - \tilde{H}(G(\tau))$, we have

1. $\Delta H(\tau) \geq 0$;
2. $\Delta H(\bar{v}) = 0$;
3. $\Delta H(\tau)$ decreases in $\tau$ when $\tau > v$.

Let $\tilde{h}(q)$ denote the derivative of $\tilde{H}(q)$, i.e., $\tilde{h}(q) = \tilde{H}'(q)$. Since $\tilde{H}$ is a convex function, $\tilde{h}$ must be weakly increasing in $q$. Define $\tilde{u}(v, \tau) = \tilde{h}(G(\tau))$ as the regularized
virtual value function. From the convexification process, we know that \( \tilde{u}(v, \tau) \geq 0, \forall \tau \) if \( \int_{0}^{\tilde{v}} u(v, \tau) dG(\tau) \geq 0 \), and \( \tilde{u}(v, \tau) < 0, \forall \tau \) if \( \int_{0}^{\tilde{v}} u(v, \tau) dG(\tau) < 0 \).

Let \( \omega(v) = \int_{0}^{\tilde{v}} u(v, \tau) dG(\tau) \). Differentiating \( \omega(v) \) with respect to \( v \), we have

\[
\omega'(v) = \lambda'(v) \left( 1 - G(v) \right) + \frac{1 - F(v)}{f(v)} g(v).
\]

With monotone hazard rate, we have \( \lambda'(v) \geq 0 \), which implies that \( \omega'(v) > 0 \), i.e., \( \omega(v) \) is weakly increasing. Since \( \omega(v) \) is continuous in \( v \), \( \omega(0) = \lambda(0) < 0 \), and \( \omega(v^*) = \int_{0}^{v^*} \tau dG(\tau) > 0 \) where \( v^* \) is such that \( \lambda(v^*) = 0 \), by intermediate value theorem, we have the following result.

**Lemma 3.** There exists a unique \( v^{**} \in [0, v^*] \) such that \( \omega(v^{**}) = 0 \).

To identify the optimal second-stage allocation rule, we follow the Myersonian procedure and have

\[
\int_{0}^{\tilde{v}} p_2(v, \tau) u(v, \tau) dG(\tau) = \int_{0}^{\tilde{v}} p_2(v, \tau) \tilde{u}(v, \tau) dG(\tau) + \int_{0}^{\tilde{v}} p_2(v, \tau) \left( \hat{h}(G(\tau)) - \tilde{h}(G(\tau)) \right) dG(\tau) = \int_{0}^{\tilde{v}} p_2(v, \tau) \tilde{u}(v, \tau) dG(\tau) - \int_{0}^{\tilde{v}} \left( H(G(\tau)) - \tilde{H}(G(\tau)) \right) dp_2(v, \tau).
\]
For any \( p_2(v, \tau) \) satisfying condition (3.5), it follows that
\[
\int_0^\bar{v} \left( H(G(\tau)) - \tilde{H}(G(\tau)) \right) dp_2(v, \tau) \\
= \int_0^v \left( H(G(\tau)) - \tilde{H}(G(\tau)) \right) dp_2(v, \tau) + \int_v^\bar{v} \left( H(G(\tau)) - \tilde{H}(G(\tau)) \right) dp_2(v, \tau) \\
= \int_0^\bar{v} \Delta H(\tau) dp_2(v, \tau) - \Delta H(\tau) p_2(v, \tau) - \int_v^\bar{v} p_2(v, \tau) d\Delta H(\tau).
\]

From Lemma 2, the decreasing property of \( \Delta H(\tau) \) in \( \tau \) as \( \tau > v \) implies
\[
\int_v^\bar{v} p_2(v, \tau) d\Delta H(\tau) \leq p_2(v, v)(\Delta H(\bar{v}) - \Delta H(v)).
\]

Therefore, we have
\[
\int_0^\bar{v} \left( H(G(\tau)) - \tilde{H}(G(\tau)) \right) dp_2(v, \tau) \\
\geq \int_0^\bar{v} \Delta H(\tau) dp_2(v, \tau) - \Delta H(\tau) p_2(v, \tau) - p_2(v, v)(\Delta H(\bar{v}) - \Delta H(v)) \\
= \int_0^\bar{v} \Delta H(\tau) dp_2(v, \tau) \geq 0,
\]
which implies
\[
\int_0^\bar{v} p_2(v, \tau) u(\tau) dG(\tau) \leq \int_0^\bar{v} p_2(v, \tau) \bar{u}(\tau) dG(\tau). \tag{3.7}
\]

On the other hand, for a given first-stage report \( v \), define \( p_2^{**}(v, \cdot) : [0, \bar{v}] \to \mathbb{R} \) as follows:
\[
p_2^{**}(v, \tau) = \begin{cases} 1 & \text{if } v \geq v^{**}, \\ 0 & \text{if } v < v^{**}, \end{cases}
\]
for all \( v \) and \( \tau \). As \( p_2^{**}(v, \tau) \) assigns winning probability 1 to types with \( \bar{u}(v, \tau) \geq 0 \). Thus, for any \( p_2(v, \tau) \) we have
\[
\int_0^\bar{v} p_2(v, \tau) \bar{u}(v, \tau) dG(\tau) \leq \int_0^\bar{v} p_2^{**}(v, \tau) \bar{u}(v, \tau) dG(\tau), \forall p_2(v, \tau). \tag{3.8}
\]

Also, by definition of \( p_2^{**}(v, \tau) \), we have \( \frac{\partial p_2^{**}(v, \tau)}{\partial \tau} = 0 \) almost everywhere, which leads to
\[
\int_0^\bar{v} \left( H(G(\tau)) - \tilde{H}(G(\tau)) \right) dp_2^{**}(v, \tau) = 0,
\]
that is,
\[
\int_0^\bar{v} p_2^{**}(v, \tau) \bar{u}(v, \tau) dG(\tau) = \int_0^\bar{v} p_2^{**}(v, \tau) u(v, \tau) dG(\tau). \tag{3.9}
\]

Combine inequality (3.7), (3.8), and (3.9), for any \( p_2(v, \tau) \) satisfying condition (3.5), we have
\[
\int_0^\bar{v} p_2(v, \tau) u(v, \tau) dG(\tau) \leq \int_0^\bar{v} p_2^{**}(v, \tau) u(v, \tau) dG(\tau).
\]
By definition, \( p^{**}_{2}(v, \tau) \) is independent of \( \tau \), so condition (3.5) is satisfied. Therefore, \( p^{**}_{2}(v, \tau) \) maximizes \( \int_{0}^{\bar{v}} p_{2}(v, \tau)u(v, \tau)dG(\tau) \) subject to condition (3.5).

We next derive the optimal \( p^{**}_{1}(v) \). With second-stage allocation \( p^{**}_{2}(v, \tau) \), the seller’s expected revenue is further written as

\[
R = \int_{0}^{v^{**}} \left\{ p_{1}(v)\lambda(v) + (1 - p_{1}(v))\left( G(v)\lambda(v) - \int_{0}^{v} \tau dG(\tau) \right) \right\} dF(v) + \int_{v^{**}}^{\bar{v}} \lambda(v)dF(v).
\]

Rearranging the above expression renders

\[
R = \int_{0}^{v^{**}} \left\{ G(v)\lambda(v) - \int_{0}^{v} \tau dG(\tau) + p_{1}(v)\omega(v) \right\} dF(v) + \int_{v^{**}}^{\bar{v}} \lambda(v)dF(v).
\]

Since \( \omega(v) < 0 \) when \( v < v^{**} \), it is optimal to make \( p^{**}_{1}(v) = 0 \) under this circumstance. For \( v \geq v^{**} \), the value of \( p_{1}(v) \) does make a difference. Thus, we can let \( p^{**}_{1}(v) = 1 \) for \( v \geq v^{**} \).

Therefore, the following first-stage allocation rule is optimal:

\[
p^{**}_{1}(v) = \begin{cases} 1 & \text{if } v \geq v^{**}, \\ 0 & \text{if } v < v^{**}. \end{cases}
\]

We summarize the above results in the following lemma.

**Lemma 4.** The above defined allocation rule \( (p^{**}_{1}(v), p^{**}_{2}(v, \tau)) \) maximizes revenue \( R \) in (3.6) subject to condition (3.5). At optimum, we have

\[
R^{*} = \int_{0}^{v^{**}} \left\{ G(v)\lambda(v) - \int_{0}^{v} \tau dG(\tau) + \omega(v) \right\} dF(v) + \int_{v^{**}}^{\bar{v}} \lambda(v)dF(v).
\]

**Proof.** See Appendix. \( \square \)

As pointed out earlier, \( R^{*} \) provides an upper bound for revenue achievable in the original setting with private \( v \) and \( \tau \). Therefore, if we can identify an incentive compatible and individual rational mechanism in the original setting, which generates revenue \( R^{*} \), then this mechanism must be optimal. The following proposition provides such a mechanism.\(^{13}\)

**Proposition 2.** In the original setting, in which both \( v \) and \( \tau \) are the buyer’s private information, the following mechanism is incentive compatible and individual rational. It achieves the revenue bound \( R^{*} \) in Lemma 4, and thus is optimal in the original setting.

\(^{13}\text{Under this single-stage mechanism, the second-stage incentive compatibility condition and individual rationality condition trivially hold.}\)
1. if $v \geq v^{**}$, the type-$v$ buyer purchases without search at price $E_{\tau} \min\{v^{**}, \tau\}$;

2. if $v < v^{**}$, the type-$v$ buyer goes for the outside option for sure.

Proof. See Appendix.

Proposition 2 reveals that when information in both stages is privately observed by the buyer, whether he gets the object from the seller only depends on his valuation. When the buyer's valuation is high ($v \geq v^{**}$), the object is sold to him at price $E_{\tau} \min\{v^{**}, \tau\}$. This means that the mechanism fully deters the search of higher types of buyers, even when there is no search cost. When the valuation is low ($v < v^{**}$), he never gets the object from the seller and goes for his outside option when the outside option price is lower than his value. Essentially, the optimal two-stage mechanism degenerates to a single-stage mechanism, while the buyer’s second-stage private information is not used.

One interesting question is why the optimal mechanism degenerates to a single-stage one and the seller never sells to a low-valuation buyer. The intuition goes as follows. Suppose the seller sells to a low-valuation ($v < v^{**}$) buyer at either the first or the second stage. There are two effects. First, the seller could gain from additional trade. Second, this would lead to higher informational rent to the high types ($v \geq v^{**}$). Since in any case, the surplus that can be extracted from the low types is rather limited, the trade-off favors shutting them down completely.

Comparing across optimal mechanisms in the benchmark setting with public second-stage information and the original setting with private second-stage information reveals the differences in allocation rules. While the optimal mechanism must be implemented in two stages in the benchmark setting, in the original setting with private second-stage information the optimal mechanism can be implemented in a single stage.

Note that by Lemma 3, we have $v^{**} < v^*$, i.e. first stage trade occurs more often when buyer option is private. The intuition behind this result is as follows. With private option $\tau$ the seller only collects payment from the high types of buyer in the first stage, while when $\tau$ is public, she also collects payment from the low types of buyer through second stage price match. Therefore, the seller should lower the value cutoff for first stage trade to increase the chance for collecting payment from the buyer at the first stage.

In the benchmark setting, high types ($v \geq v^*$) buy the object from the seller at price $E_{\tau} \min\{v^*, \tau\}$; in the original setting, high types ($v \geq v^*$) buy the object from the seller at
price \( E\tau \min\{v^{**}, \tau\} \). Recall \( v^{**} < v^* \). Thus, high types \((v \geq v^*)\) benefit from the privateness of the outside option price. Types \( v \in (v^{**}, v^*) \) get the object from the seller for sure in the first stage in the original setting with private \( \tau \). But in the benchmark setting with public \( \tau \) they only get the object from the seller in the second stage if \( \tau \) turns out to be lower than \( v \). In the original setting the buyer always pays \( E\tau \min\{v^{**}, \tau\} < v \), which is lower than the payment in the benchmark setting (i.e., \( E_{\tau \leq v}\tau\)).\footnote{Note that payment of types \( v \in (v^{**}, v^*) \) in the benchmark setting does not depend on whether the buyer obtains the object in the second stage from the seller or the outside option.} Thus, these types also benefit from the privateness of the outside option price: with private option \( \tau \), these types enjoy higher trade surplus. Low types \((v < v^{**})\) have the same expected payoffs in both settings, as they only get the object when \( \tau \) turns out to be lower than \( v \) and pay the same. In the original setting, the buyer gets the object from the outside option; while in the benchmark setting, he gets it from the seller. Due to the price match feature of the optimal selling mechanism for the benchmark setting, the payment across the two environments is exactly the same.

Recall that \( R^* \) is the revenue achieved in the benchmark setting while imposing semi-monotonicity constraint \((3.5)\). Therefore, the optimal revenue with private \( \tau \) is strictly lower than that in the benchmark setting with public \( \tau \). In other words, the privateness of the buyer’s outside option hurts the seller.

We summarize these results in the following corollary.

**Corollary 1.** Compared to the benchmark environment in which the buyer’s outside option is public, the privateness of the outside option price benefits the buyer and hurts the seller.

Corollary 1 reveals that the privateness of the agent’s ex post information can make a difference in dynamic mechanism design, even when both the agent’s ex ante and ex post information are continuously distributed. Esö and Szentes (2007, 2017) find that in a fairly wide range of contexts with continuously distributed private information, the privateness of ex post (orthogonalized) information is irrelevant to the optimal design and the players’ payoffs. Krähmer and Strausz (2015b) consider two scenarios: one with discrete ex ante information and one with continuous ex ante information. They find that the irrelevance of ex post (orthogonalized) information holds if and only if ex ante information is continuously distributed. Kakade, Lobel, and Nazerzadeh (2013) and Battaglini and Lamba (2019) demonstrate environments with continuous types, in which the irrelevance of privateness
of ex post information would fail. Our model further provides an example environment in which the irrelevance result fails when both ex ante and ex post information are continuously distributed.\textsuperscript{15} Our analysis reveals the reason for this divergence in our model: the optimal allocation rule in the benchmark setting with public ex post information fails to satisfy the necessary semi-monotonicity condition required by second-stage incentive compatibility in the original setting with private ex post information.

Differentiating from the above mentioned studies which do not identify the optimal mechanisms when the irrelevance fails, we are able to fully characterize the optimal mechanism in our environment. Identifying the optimal mechanism in our model with private ex post information requires the application of a convexification procedure first coined by Myerson (1981), who considers a single-stage optimal auction design problem. To our best knowledge, our paper is the first study that exemplifies its validity when being applied to ex post information in a dynamic environment with the additional complication of potentially non-monotone second stage allocation rules.

4 Conclusion

In this paper, we study the optimal two-stage selling mechanism in an environment in which the buyer’s outside optional price arrives costlessly at the second stage. We consider cases that allow second-stage information to be either public or private.

We find that in our model the privateness of second-stage information makes a difference in the optimal design. Our model is endowed with unique features, which calls for innovative analysis. With a private outside price, second-stage incentive compatibility imposes binding semi-monotonicity conditions on feasible allocation rules, and off-equilibrium-path second stage best strategy cannot be pinned down. As a result, deriving the optimal mechanism with private buyer option requires a new method, which replies on an adapted Myerson convexification procedure to regularize the buyers’ virtual values in the dimension of the outside prices. To our best knowledge, this is the first time in the literature the procedure is employed in a dynamic mechanism design setting. While this approach is discovered in a particular setting, the procedure could be useful for addressing other problems with similar issues.

\textsuperscript{15}In our model, ex post information is independent of ex ante information by construction.
With private option, the seller never sells to the buyer if his value is lower than a threshold. The optimal selling mechanism takes a simple form of first stage fixed price. With buyer option being public information, we find that a second stage price matching is necessarily a feature of the optimal mechanism. The seller either sells at a fixed price immediately to the buyer if his value is above a bar; or if the buyer’s value is lower than the bar, he goes for search, and the seller will match the deal the buyer will find.

In this paper, we focus on an environment where seller has commitment power. A natural and interesting extension is to consider the optimal design when seller has no commitment power. We leave this to future work.
5 Appendix

Proof of Proposition 1: Rewrite the seller’s expected revenue as

$$R = \int_{0}^{\bar{v}} \left\{ -\pi(v) + p_{1}(v)v + (1 - p_{1}(v))E_{\tau}(\bar{\pi}(v, \tau) + x_{2}(v, \tau)) \right\} dF(v).$$

Recall that \(\bar{\pi}(v, \tau) = \max\{v - \tau, 0\} + p_{2}(v, \tau)\min\{v, \tau\} - x_{2}(v, \tau)\), and the seller’s revenue is rewritten as

$$R = \int_{0}^{\bar{v}} \left\{ -\pi(v) + p_{1}(v)v + (1 - p_{1}(v))E_{\tau}\left(\max\{v - \tau, 0\} + p_{2}(v, \tau)\min\{v, \tau\}\right) \right\} dF(v).$$

Since \(\int_{0}^{\bar{v}} \pi(v) dF(v) = \pi(0) + \int_{0}^{\bar{v}} \pi'(v)(1 - F(v))/f(v) dF(v)\), we have

$$R = -\pi(0) + \int_{0}^{\bar{v}} \left\{ -\frac{1 - F(v)}{f(v)}\pi'(v) + p_{1}(v)v \right\} + (1 - p_{1}(v))E_{\tau}\left(\max\{v - \tau, 0\} + p_{2}(v, \tau)\min\{v, \tau\}\right) dF(v).$$

By the envelope theorem, \(d\pi(v)/dv = \partial\pi(v, v')/\partial v\) at \(v' = v\). Then we get

$$\pi'(v) = p_{1}(v) + (1 - p_{1}(v))[G(v) + \int_{v}^{\bar{v}} p_{2}(v, \tau)dG(\tau)].$$

Substitute it into \(R\),

$$R = -\pi(0) + \int_{0}^{\bar{v}} \left\{ p_{1}(v)\lambda(v) + (1 - p_{1}(v))[G(v)\lambda(v) - \int_{0}^{v} \tau dG(\tau) \right\}$$

$$+ \int_{v}^{\bar{v}} p_{2}(v, \tau)\tau dG(\tau) + \int_{v}^{\bar{v}} p_{2}(v, \tau)\lambda(v)dG(\tau) \right\} dF(v).$$

The seller maximizes \(R\) by choosing \(\{p_{1}(v), p_{2}(v, \tau)\}\) subject to constraints (3.1) and (3.2). Since from (3.2) we learn that \(\pi(0) \geq 0\), to obtain the optimum value of \(R\) we should make \(\pi(0) = 0\). That is, a buyer with value 0 gets expected payoff 0 at the first stage. Then the maximization problem is

$$\max_{\{p_{1}(v), p_{2}(v, \tau)\}} R = \int_{0}^{\bar{v}} \left\{ p_{1}(v)\lambda(v) + (1 - p_{1}(v))[G(v)\lambda(v) - \int_{0}^{v} \tau dG(\tau) \right\}$$

$$+ \int_{v}^{\bar{v}} p_{2}(v, \tau)\tau dG(\tau) + \int_{v}^{\bar{v}} p_{2}(v, \tau)\lambda(v)dG(\tau) \right\} dF(v).$$

We first consider the second-stage mechanism. Given value \(v\), the second-stage allocation rule \(p_{2}(v, \tau)\) should maximize

$$\int_{0}^{v} p_{2}(v, \tau)\tau dG(\tau) + \int_{v}^{\bar{v}} p_{2}(v, \tau)\lambda(v)dG(\tau).$$
From pointwise maximization, it is optimal to set

\[ p_2^*(v, \tau) = \begin{cases} 
0 & \text{if } \tau > v \text{ and } v < v^*, \\
1 & \text{otherwise}. 
\end{cases} \]

Allocation rule \( p_2^*(v, \tau) \) says that the object should not be sold to a buyer with low value and a bad outside option.

Next, we consider the corresponding second-stage payment rule. By the individual rationality constraint, the optimal second-stage payment rule should be such that

\[
\max \{v - \tau, 0\} + p_2^*(v, \tau) \min \{v, \tau\} - x_2^*(v, \tau) \geq \max \{v - \tau, 0\}.
\]

It implies \( x_2^*(v, \tau) \leq p_2^*(v, \tau) \min \{v, \tau\} \). With \( \tau \) being publically observed, the seller thus sets the payment to make the agent to be indifferent. That is, \( x_2^*(v, \tau) = p_2^*(v, \tau) \min \{v, \tau\} \).

With second-stage allocation rule \( p_2^*(v, \tau) \), the seller’s expected revenue is

\[
R = \int_0^{v^*} \left\{ p_1(v)\lambda(v) + (1 - p_1(v))G(v)\lambda(v) \right\} dF(v) \\
+ \int_{v^*}^\vartheta \left\{ p_1(v)\lambda(v) + (1 - p_1(v))\left[G(v)\lambda(v) + \lambda(v)(1 - G(v))\right] \right\} dF(v),
\]

which is simplified as

\[
R = \int_0^{v^*} \left\{ G(v)\lambda(v) + p_1(v)\lambda(v)(1 - G(v)) \right\} dF(v) + \int_{v^*}^\vartheta \lambda(v)dF(v).
\]

To maximize \( R \), the seller could set the first-stage allocation rule as

\[ p_1^*(v) = \begin{cases} 
0 & \text{if } v < v^*, \\
1 & \text{otherwise.} 
\end{cases} \]

The corresponding expected revenue for the seller is

\[
R^* = \int_0^{v^*} G(v)\lambda(v)dF(v) + \int_{v^*}^\vartheta \lambda(v)dF(v).
\]

Consider the first-stage payment rule

\[ x_1^*(v) = \begin{cases} 
0 & \text{if } v < v^*, \\
E_\tau \min \{v^*, \tau\} & \text{otherwise}. 
\end{cases} \]

One can verify that mechanism \( \mathcal{M} = \{p_1^*, x_1^*; p_2^*, x_2^*\} \) indeed satisfies constraints (3.1) and (3.2). Moreover, it achieves the seller revenue \( R^* \). ||
Proof of Lemma 1: Rewrite the buyer’s second-stage expected payoff when he has outside option price \( \tau \), but reports \( \tau' \) after a truthful first stage as follows:

\[
\tilde{\pi}(v, \tau, \tau') = \tilde{\pi}(v, \tau') + (1 - p_2(v, \tau')) \left( \min\{v, \tau'\} - \min\{v, \tau\} \right).
\]

The second-stage incentive compatibility condition \((IC_2)\) is equivalent to

\[
\tilde{\pi}(v, \tau) \geq \tilde{\pi}(v, \tau') + (1 - p_2(v, \tau')) \left( \min\{v, \tau'\} - \min\{v, \tau\} \right).
\]

(5.1)

Thus, a two-stage mechanism is incentive compatible in the second stage upon truthful report in the first stage if and only if (5.1) holds.

Switching the roles of \( \tau \) and \( \tau' \) in (5.1), we get

\[
\tilde{\pi}(v, \tau') \geq \tilde{\pi}(v, \tau) + (1 - p_2(v, \tau)) \left( \min\{v, \tau\} - \min\{v, \tau'\} \right).
\]

Let \( \Delta(v, \tau, \tau') = \min\{v, \tau'\} - \min\{v, \tau\} \); then second-stage incentive compatibility implies that

\[
(1 - p_2(v, \tau')) \Delta(v, \tau, \tau') \leq \tilde{\pi}(v, \tau) - \tilde{\pi}(v, \tau') \leq (1 - p_2(v, \tau)) \Delta(v, \tau, \tau').
\]

For \( \tau' < \tau \), we have \( \Delta(v, \tau, \tau') < 0 \) when \( 0 \leq \tau' < v \). Then (3.5) follows.

When \( \tau' \geq v \) and \( \tau \geq v \), we have \( \Delta(v, \tau, \tau') = 0 \), and \( \tilde{\pi}(v, \tau) = \tilde{\pi}(v, \tau') \). Thus, it is unclear how \( p_2(v, \tau) \) changes when \( \tau \geq v \).

Proof of Lemma 2: By the convexification process, \( \Delta H(\tau) \geq 0 \) and \( \Delta H(\bar{v}) = 0 \) follow directly, and we have the convex hull

\[
\hat{H}(q) = \begin{cases} 
\int_0^q G^{-1}(s)ds & \text{if } q \leq q_1, \\
\int_0^{q_1} G^{-1}(s)ds + G^{-1}(q_1)(q - q_1) & \text{if } q > q_1,
\end{cases}
\]

in which the tangent point \( q_1 < G(v) \) and it satisfies

\[
H(G(v)) + (1 - G(v))\lambda(v) = H(q_1) + (1 - q_1)G^{-1}(q_1).
\]

(5.2)

Since \( H(q) \) is convex when \( q \leq G(v) \), we have

\[
H(G(v)) - H(q_1) \geq (G(v) - q_1)H'(q_1) = (G(v) - q_1)G^{-1}(q_1).
\]

On the other hand, by equation (5.2) it follows that

\[
H(G(v)) - H(q_1) = (1 - q_1)G^{-1}(q_1) - (1 - G(v))\lambda(v).
\]
Thus, we obtain
\[ (1 - q_1)G^{-1}(q_1) - (1 - G(v))\lambda(v) \geq (G(v) - q_1)G^{-1}(q_1), \]
which implies \( \lambda(v) \leq G^{-1}(q_1) \).

When \( \tau \geq v \), the difference is
\[ H(q) - \bar{H}(q) = \int_{q_1}^{G(v)} G^{-1}(s)ds + \lambda(v)\left( q - G(v) \right) - G^{-1}(q_1)\left( q - q_1 \right). \]
Then we have
\[ \frac{\partial}{\partial q} \left( H(q) - \bar{H}(q) \right) = \lambda(v) - G^{-1}(q_1) \leq 0. \]
Thus, the difference \( H(q) - \bar{H}(q) \) is decreasing in \( q \). Since \( q = G(\tau) \) increases in \( \tau \), we have \( \Delta H(\tau) = H(G(\tau)) - \bar{H}(G(\tau)) \) decreases in \( \tau \) when \( \tau \geq v \).

**Proof of Lemma 4:** Recall that the seller’s expected revenue (3.6) is written as
\[ R = \int_0^\bar{v} \left\{ p_1(v)\lambda(v) + (1-p_1(v))\left[ G(v)\lambda(v) - \int_0^v \tau dG(v) + \int_0^\bar{v} p_2(v,\tau)u(v,\tau)dG(\tau) \right] \right\}dF(v). \]

We have shown that given \( v \), for \( \forall p_2(v,\tau) \),
\[ \int_0^\bar{v} p_2(v,\tau)u(v,\tau)dG(\tau) \leq \int_0^\bar{v} p_2^*(v,\tau)u(v,\tau)dG(\tau). \]
Since \( 1 - p_1(v) \geq 0 \) for \( \forall p_1(v) \), we have
\[
R \leq \int_0^\bar{v} \left\{ p_1(v)\lambda(v) + (1-p_1(v))\left[ G(v)\lambda(v) - \int_0^v \tau dG(v) + \int_0^\bar{v} p_2^*(v,\tau)u(v,\tau)dG(\tau) \right] \right\}dF(v)
= \int_0^{v^*} \left\{ G(v)\lambda(v) - \int_0^v \tau dG(\tau) + p_1(v)\omega(v) \right\}dF(v) + \int_{v^*}^\bar{v} \lambda(v)dF(v)
\leq \int_0^{v^*} \left\{ G(v)\lambda(v) - \int_0^v \tau dG(\tau) + p_1^*(v)\omega(v) \right\}dF(v) + \int_{v^*}^\bar{v} \lambda(v)dF(v)
= \int_0^{\bar{v}} \left\{ G(v)\lambda(v) - \int_0^\bar{\tau} dG(\tau) + \omega(v) \right\}dF(v) + \int_{v^*}^\bar{v} \lambda(v)dF(v). \]

By definition of \( p_2^*(v,\tau) \), it is obvious that condition (3.5) is satisfied. Thus, Lemma 4 is proved. ||

**Proof of Proposition 2:** First, we show that the proposed mechanism in Proposition 2 indeed achieves the revenue upper bound in Lemma 4.
From the proof of Lemma 4, we have the upper bound of revenue

\[ R = \int_0^{v^*} \{ G(v) \lambda(v) - \int_0^v \tau dG(\tau) \} dF(v) + \int_0^{\bar{v}} \lambda(v) dF(v) \]

\[ = - \left[ \int_0^{v^*} \int_0^{v^*} u(v, \tau) dF(v) dG(\tau) + \int_0^{\bar{v}} \int_0^{v^*} u(v, \tau) dF(v) dG(\tau) \right] + \int_0^{\bar{v}} \lambda(v) dF(v). \]

The last part of the expression above equals 0, since

\[ \int_0^{\bar{v}} \lambda(v) dF(v) = \int_0^{\bar{v}} v - 1 - F(v) dF(v) \]

\[ = - \int_0^{\bar{v}} v dF(v) - \int_0^{\bar{v}} 1 - F(v) dv \]

\[ = - \int_0^{\bar{v}} v d(1 - F(v)) - \int_0^{\bar{v}} 1 - F(v) dv \]

\[ = - v(1 - F(v)) |^{\bar{v}}_0 + \int_0^{\bar{v}} 1 - F(v) dv - \int_0^{\bar{v}} 1 - F(v) dv \]

\[ = 0. \]

Then it follows that

\[ \bar{R} = - \int_0^{\bar{v}} \int_0^{v^*} u(v, \tau) dF(v) dG(\tau) \]

\[ = (1 - F(v^*)) \int_0^{\bar{v}} \min\{\tau, v^*\} dG(\tau) = (1 - F(v^*)) \tau \min\{\tau, v^*\}. \]

On the other hand, with the mechanism stated in Proposition 2, only the buyer with valuation \( v \geq v^* \) will make a purchase at the fixed price \( E_\tau \min\{\tau, v^*\} \), which gives the seller’s expected revenue

\[ R = (1 - F(v^*)) E_\tau \min\{\tau, v^*\}. \]

Thus, the upper bound of revenue is indeed achieved.

Next, we show that the mechanism proposed in Proposition 2 is feasible. With this mechanism, a buyer’s expected payoff is

\[ \pi(v) = \begin{cases} 
  v - E_\tau \min\{v^*, \tau\} & \text{if } v \geq v^*, \\
  E_\tau \max\{v - \tau, 0\} & \text{if } v < v^*.
\end{cases} \]

It is obvious that \( \pi(v) \geq E_\tau \max\{v - \tau, 0\} \) for any buyer type \( v \). Thus, the individual rationality condition is satisfied.

If a buyer with value \( v \geq v^* \) deviates and goes for the outside option, his expected deviation payoff is \( E_\tau \max\{v, \tau\} \). Since

\[ v - E_\tau \min\{v^*, \tau\} \geq v - E_\tau \min\{v, \tau\} = E_\tau \max\{v, \tau\}, \]

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his deviation payoff is smaller than his original expected payoff.

If a buyer with value $v < v^{**}$ deviates and purchases at price $E_\tau \min\{v^{**}, \tau\}$, his deviation payoff is $v - E_\tau \min\{v^{**}, \tau\}$. Since

$$v - E_\tau \min\{v^{**}, \tau\} \leq v - E_\tau \min\{v, \tau\} = E_\tau \max\{v, \tau\},$$

his deviation payoff now is less than not purchasing from the seller, as instructed by the mechanism. Thus, the mechanism proposed in Proposition 2 is incentive compatible. ||
References


